Lineare Substitution: (Die innere Funktion ist linear, deren Ableitung konstant)

$$\int_0^1 3e^{-2x}\,dx = \int_0^1 -\tfrac{3}{2}e^{z}\,dz = \left[-\tfrac{3}{2}e^{z}\right]_{z(0)}^{z(1)} = \left[-\tfrac{3}{2}e^{-2x}\right]_0^1 = -\tfrac{3}{2}e^{-2} + \tfrac{3}{2}e^{0} = -\frac{3}{2e^2} + 1,5$$

Substitution:	$z(x) = -2x$	
Ableitung:	$\frac{dz}{dx} = -2$	$\mid \cdot dx \mid : (-2)$
Nach dx auflösen:	$\frac{dz}{-2} = dx$	

Allgemeine Substitution: (Funktioniert hier nur, weil sich das x wegkürzt!)

$$\int_2^3 \frac{5x}{x^2-3}\,dx = \int_2^3 \frac{5x}{z}\cdot\frac{dz}{2x} = \int_2^3 \frac{5}{2}\cdot\frac{dz}{z} = \left[\tfrac{5}{2}\ln|z|\right]_{z(2)}^{z(3)} = \left[\tfrac{5}{2}\ln|x^2-3|\right]_2^3 = \tfrac{5}{2}\ln 6 - \tfrac{5}{2}\ln 1 = \tfrac{5}{2}\ln 6$$

Substitution:	$z(x) = x^2 - 3$	
Ableitung:	$\frac{dz}{dx} = 2x$	$\mid \cdot dx \mid : (2x)$
Nach dx auflösen:	$\frac{dz}{2x} = dx$	

Nach der Substitution kommen nur folgende Integrale vor:

- $\displaystyle\int_a^b c\cdot z^k\,dz = \left[c\cdot\tfrac{1}{k+1}z^{k+1}\right]_{z(a)}^{z(b)}$ (mit $k \neq -1$)
- $\displaystyle\int_a^b c\cdot e^z\,dz = \left[c\cdot e^z\right]_{z(a)}^{z(b)}$
- $\displaystyle\int_a^b c\cdot\frac{dz}{z} = \left[c\cdot\ln|z|\right]_{z(a)}^{z(b)}$ (s.o. mit $k=-1$)
- $\displaystyle\int_a^b c\cdot\sin(z)\,dz = \left[-c\cdot\cos(z)\right]_{z(a)}^{z(b)}$

Aufgaben: Berechne die Integrale mit der Substitutionsmethode.

a) $\displaystyle\int_0^{\frac{1}{3}} \frac{7}{3x-2}\,dx$ b) $\displaystyle\int_1^2 3x(x^2-4)^4\,dx$ c) $\displaystyle\int_{-1}^0 2e^x\cdot\sqrt{3e^x+1}\,dx$

d) $\displaystyle\int_0^2 \frac{-e^x}{\sqrt{e^x+3}}\,dx$ e) $\displaystyle\int_2^4 \frac{-1}{\left(4-\frac{1}{2}x\right)^2}\,dx$ f) $\displaystyle\int_{-4}^2 x\cdot e^{-\frac{1}{2}x^2+1}\,dx$

g) $\displaystyle\int_0^1 \frac{5e^{2x}}{e^{2x}+3}\,dx$ h) $\displaystyle\int x\cdot\sin\left(4x^2+\pi\right)\,dx$ i) $\displaystyle\int \sqrt[3]{\sin(4x)}\cdot\cos(4x)\,dx$

Folgende Lösungen kommen hier vor:

$z(x)$	Integrand	Stammfunktion	Ergebnis
x^2-4	$c\cdot z^{-\frac{1}{2}}$	$-2\sqrt{e^x+3}$	$-\frac{1}{3}\approx -0,333$
$3e^x+1$	$c\cdot\sqrt{z}$	$\frac{7}{3}\ln|3x-2|$	$4-2\sqrt{e^2+3}\approx -2,446$
$\sin(4x)$	$c\cdot z^4$	$-e^{-\frac{1}{2}x^2+1}$	$\frac{32}{9}-\frac{4}{9}(\frac{3}{e}+1)^{1,5}\approx 2,200$
$4-\frac{1}{2}x$	$c\cdot z^{\frac{1}{3}}$	$-\frac{2}{4-\frac{1}{2}x}$	$\frac{729}{10}=72,9$
$-\frac{1}{2}x^2+1$	$c\cdot\frac{1}{z}$	$\frac{5}{2}\ln|e^{2x}+3|$	$e^{-7}-e^{-1}\approx -0,367$
$4x^2+\pi$	$c\cdot e^z$	$\frac{4}{9}(3e^x+1)^{1,5}$	$\frac{5}{2}\left(\ln(e^2+3)-\ln 4\right)\approx 2,386$
$e^{2x}+3$	$c\cdot z^{-2}$	$-\frac{1}{8}\cos(4x^2+\pi)$	$-\frac{7}{3}\ln 2\approx -1,617$
e^x+3	$c\cdot\frac{1}{z}$	$\frac{3}{10}(x^2-4)^5$	
$3x-2$	$c\cdot\sin(z)$	$\frac{3}{16}(\sin(4x))^{\frac{4}{3}}$	

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Aufgaben: Berechne die Integrale mit der Substitutionsmethode.

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e) $\int_2^4 \frac{-1}{\left(4-\frac{1}{2}x\right)^2}\,dx$

f) $\int_{-4}^2 x\cdot e^{-\frac{1}{2}x^2+1}\,dx$

g) $\int_0^1 \frac{5e^{2x}}{e^{2x}+3}\,dx$

h) $\int x\cdot\sin\left(4x^2+\pi\right)\,dx$

i) $\int \sqrt[3]{\sin(4x)}\cdot\cos(4x)\,dx$

Folgende Lösungen kommen hier vor:

$z(x)$	Integrand	Stammfunktion	Ergebnis
x^2-4 b)	$c\cdot z^{-\frac{1}{2}}$ d)	$-2\sqrt{e^x+3}$ d)	$-\frac{1}{3}\approx -0,333$ e)
$3e^x+1$ c)	$c\cdot\sqrt{z}$ c)	$\frac{7}{3}\ln\|3x-2\|$ a)	$4-2\sqrt{e^2+3}\approx -2,446$ d)
$\sin(4x)$ i)	$c\cdot z^4$ b)	$-e^{-\frac{1}{2}x^2+1}$ f)	$\frac{32}{9}-\frac{4}{9}(\frac{3}{e}+1)^{1,5}\approx 2,200$ c)
$4-\frac{1}{2}x$ e)	$c\cdot z^{\frac{1}{3}}$ i)	$-\frac{2}{4-\frac{1}{2}x}$ e)	$\frac{729}{10}=72,9$ b)
$-\frac{1}{2}x^2+1$ f)	$c\cdot\frac{1}{z}$ a)	$\frac{5}{2}\ln\|e^{2x}+3\|$ g)	$e^{-7}-e^{-1}\approx -0,367$ f)
$4x^2+\pi$ h)	$c\cdot e^z$ f)	$\frac{4}{9}(3e^x+1)^{1,5}$ c)	$\frac{5}{2}(\ln(e^2+3)-\ln 4)\approx 2,386$ g)
$e^{2x}+3$ g)	$c\cdot z^{-2}$ e)	$-\frac{1}{8}\cos(4x^2+\pi)$ h)	$-\frac{7}{3}\ln 2\approx -1,617$ a)
e^x+3 d)	$c\cdot\frac{1}{z}$ g)	$\frac{3}{10}(x^2-4)^5$ b)	
$3x-2$ a)	$c\cdot\sin(z)$ h)	$\frac{3}{16}(\sin(4x))^{\frac{4}{3}}$ i)	

a) $$\int_0^{\frac{1}{3}} \frac{7}{3x-2}\,dx = \int_0^{\frac{1}{3}} \frac{7}{3}\cdot\frac{dz}{z} = \left[\tfrac{7}{3}\ln|z|\right] = \left[\tfrac{7}{3}\ln|3x-2|\right]_0^{\frac{1}{3}} = \tfrac{7}{3}\ln 1 - \tfrac{7}{3}\ln 2 = -\tfrac{7}{3}\ln 2 \approx -1{,}617$$

$$\boxed{z(x) = 3x-2 \quad\Longrightarrow\quad \tfrac{dz}{dx} = 3 \quad\Longrightarrow\quad dx = \tfrac{dz}{3}}$$

b) $$\int_1^2 3x\cdot(x^2-4)^4\,dx = \int_1^2 3x\cdot z^4\cdot\frac{dz}{2x} = \int_1^2 \tfrac{3}{2}z^4\,dz = \left[\tfrac{3}{2}\cdot\tfrac{1}{5}z^5\right] = \left[\tfrac{3}{10}(x^2-4)^5\right]_1^2$$

$$= \tfrac{3}{10}\cdot 0 - \tfrac{3}{10}\cdot(-3)^5 = \tfrac{3}{10}\cdot 3^5 = 72{,}9$$

$$\boxed{z(x) = x^2-4 \quad\Longrightarrow\quad \tfrac{dz}{dx} = 2x \quad\Longrightarrow\quad dx = \tfrac{dz}{2x}}$$

c) $$\int_{-1}^0 2e^x\cdot\sqrt{3e^x+1}\,dx = \int_{-1}^0 2e^x\cdot\sqrt{z}\,\frac{dz}{3e^x} = \int_{-1}^0 \tfrac{2}{3}z^{\frac{1}{2}}\,dz = \left[\tfrac{2}{3}\cdot\tfrac{1}{\frac{3}{2}}z^{1{,}5}\right] = \left[\tfrac{4}{9}(3e^x+1)^{1{,}5}\right]_{-1}^0$$

$$= \tfrac{4}{9}(3\cdot 1+1)^{1{,}5} - \tfrac{4}{9}(3e^{-1}+1)^{1{,}5} = \tfrac{4}{9}\cdot 8 - \tfrac{4}{9}(\tfrac{3}{e}+1)^{1{,}5} \approx 2{,}2$$

$$\boxed{z(x) = 3e^x+1 \quad\Longrightarrow\quad \tfrac{dz}{dx} = 3e^x \quad\Longrightarrow\quad dx = \tfrac{dz}{3e^x}}$$

d) $$\int_0^2 \frac{-e^x}{\sqrt{e^x+3}}\,dx = \int_0^2 \frac{-e^x}{\sqrt{z}}\cdot\frac{dz}{e^x} = \int_0^2 -z^{-\frac{1}{2}}\,dz = \left[-\tfrac{1}{\frac{1}{2}}z^{\frac{1}{2}}\right] = \left[-2\sqrt{e^x+3}\right]_0^2$$

$$= -2\sqrt{e^2+3} + 2\sqrt{1+3} = 4 - 2\sqrt{e^2+3} \approx -2{,}446$$

$$\boxed{z(x) = e^x+3 \quad\Longrightarrow\quad \tfrac{dz}{dx} = e^x \quad\Longrightarrow\quad dx = \tfrac{dz}{e^x}}$$

e) $$\int_2^4 \frac{-1}{\left(4-\frac{1}{2}x\right)^2}\,dx = \int_2^4 \frac{-1}{z^2}\cdot(-2)dz = \int_2^4 2z^{-2}dz = \left[2\cdot\tfrac{1}{-1}z^{-1}\right] = \left[\frac{-2}{4-\frac{1}{2}x}\right]_2^4 = \tfrac{-2}{4-2} - \tfrac{-2}{4-1} = -\tfrac{1}{3}$$

$$\boxed{z(x) = 4-\tfrac{1}{2}x \quad\Longrightarrow\quad \tfrac{dz}{dx} = -\tfrac{1}{2} \quad\Longrightarrow\quad dx = -2dz}$$

f) $$\int_{-4}^2 x\cdot e^{-\frac{1}{2}x^2+1}dx = \int_{-4}^2 x\cdot e^z\cdot\frac{dz}{-x} = \int_{-4}^2 -e^z dz = \left[-e^z\right] = \left[-e^{-\frac{1}{2}x^2+1}\right]_{-4}^2 = -e^{-1}+e^{-7} \approx -0{,}367$$

$$\boxed{z(x) = -\tfrac{1}{2}x^2+1 \quad\Longrightarrow\quad \tfrac{dz}{dx} = -x \quad\Longrightarrow\quad dx = \tfrac{dz}{-x}}$$

g) $$\int_0^1 \frac{5e^{2x}}{e^{2x}+3}\,dx = \int_0^1 \frac{5e^{2x}}{z}\cdot\frac{dz}{2e^{2x}} = \int_0^1 \frac{5}{2}\cdot\frac{dz}{z} = \left[\tfrac{5}{2}\ln|z|\right] = \left[\tfrac{5}{2}\ln|e^{2x}+3|\right]_0^1$$

$$= \tfrac{5}{2}\ln(e^2+3) - \tfrac{5}{2}\ln 4 \approx 2{,}386$$

$$\boxed{z(x) = e^{2x}+3 \quad\Longrightarrow\quad \tfrac{dz}{dx} = 2e^{2x} \quad\Longrightarrow\quad dx = \tfrac{dz}{2e^{2x}}}$$

h) $$\int x\cdot\sin(4x^2+\pi)\,dx = \int x\cdot\sin(z)\cdot\frac{dz}{8x} = \int \tfrac{1}{8}\sin(z)\,dz = -\tfrac{1}{8}\cos(z) = -\tfrac{1}{8}\cos(4x^2+\pi)$$

$$\boxed{z(x) = 4x^2+\pi \quad\Longrightarrow\quad \tfrac{dz}{dx} = 8x \quad\Longrightarrow\quad dx = \tfrac{dz}{8x}}$$

i) $$\int \sqrt[3]{\sin(4x)}\cdot\cos(4x)\,dx = \int \sqrt[3]{z}\cdot\cos(4x)\,\frac{dz}{4\cos(4x)} = \int \frac{1}{4}\cdot z^{\frac{1}{3}}\,dz = \frac{1}{4}\cdot\frac{1}{\frac{4}{3}}\cdot z^{\frac{4}{3}}$$

$$= \tfrac{3}{16}\cdot(\sin(4x))^{\frac{4}{3}}$$

$$\boxed{z(x) = \sin(4x) \quad\Longrightarrow\quad \tfrac{dz}{dx} = 4\cos(4x) \quad\Longrightarrow\quad dx = \tfrac{dz}{4\cos(4x)}}$$